Foundation ActEd Course

Comment

This is a sample of the Foundation ActEd Course (FAC), which covers the mathematical material needed as the background for Subjects CT1, CT3 to CT6 and CT8. The Institute and Faculty produce guidance on the mathematics skills required by students joining the profession. FAC reflects this guidance and therefore comprehensively covers all the mathematics required for the Core Technical subjects. Unlike other mathematical textbooks it is geared specifically towards actuarial students. The Institute and Faculty list FAC as a recommended text.

Fractional or negative powers

Very often in practice, the power on the bracket will not be a positive integer, so alternatively we can use:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

where n is negative or fractional, and $-1 < x \leq 1$.

Note that the first term in this type of series has to be 1.

Example

Expand $\sqrt{(1-2x)}$ as far as the term in x^3. For what values of x is this expansion valid?

Solution

$$\begin{aligned}\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}} &= 1 + \tfrac{1}{2}(-2x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-2x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(-2x)^3 + \cdots \\ &= 1 - x - \tfrac{1}{2}x^2 - \tfrac{1}{2}x^3 + \cdots\end{aligned}$$

This expansion is valid for $-1 < 2x \leq 1$, or $-\tfrac{1}{2} < x \leq \tfrac{1}{2}$.

Partial differentiation

If $f(x,y)$ is a function of two independent variables x and y, then the partial derivatives of $f(x,y)$ with respect to x and y are defined to be:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \qquad \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

respectively. Effectively they can be calculated by differentiating f with respect to x treating y as a constant for $\frac{\partial f}{\partial x}$, and differentiating f with respect to y treating x as a constant for $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial x}$, the partial derivative of f with respect to x, tells you the rate of change of the function f when x is varied but all other variables are kept constant.

Example

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function $f(x,y) = 2x^2y + (x+2y)^3$.

Solution

$$\frac{\partial f}{\partial x} = 4xy + 3(x+2y)^2 \qquad \frac{\partial f}{\partial y} = 2x^2 + 6(x+2y)^2$$

Higher derivatives can be found in a similar way. The notation used here is:

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^3 f}{\partial x^3}\ etc \qquad \frac{\partial^2 f}{\partial y^2}, \frac{\partial^3 f}{\partial y^3}\ etc \qquad \frac{\partial^2 f}{\partial x \partial y}\ etc$$

$\frac{\partial^2 f}{\partial x \partial y}$ means partially differentiate $\frac{\partial f}{\partial y}$ with respect to x.

Differentiating an integral (Leibniz's formula)

We have already mentioned the concept of integration being the reverse of differentiation. We will now look at differentiating an integral.

This section uses the result that $\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt$, where a and b are constants. This can be thought of as taking the $\frac{d}{dx}$ inside the integral. This can be generalised to:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = b'(x)f[x,b(x)] - a'(x)f[x,a(x)] + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt$$

This formula can be found on page 3 of the *Tables*. The proof of this result is beyond the scope of this course, but it is really just an application of the function of a function rule.

The formula is very useful in cases where the integral cannot easily be evaluated directly, for example $\frac{d}{dx}\int_0^{\infty} e^{-xt^2}\,dt$.

Example

Find $\frac{d}{dx}\int_0^x x^2 + t\,dt$.

Solution

Here $a(x) = 0$, $b(x) = x$, $f(x,t) = x^2 + t$, so:

$$\begin{aligned}\frac{d}{dx}\int_0^x x^2 + t\,dt &= 1(x^2 + x) - 0 + \int_0^x 2x\,dt \\ &= x^2 + x + 2x^2 \\ &= 3x^2 + x\end{aligned}$$

In this case we can show that this is the same as integrating directly:

$$\int_0^x x^2 + t\,dt = \left[x^2 t + \frac{1}{2}t^2\right]_0^x = x^3 + \frac{1}{2}x^2$$

So $\frac{d}{dx}\int_0^x x^2 + t\,dt = \frac{d}{dx}\left(x^3 + \frac{1}{2}x^2\right) = 3x^2 + x$.

Question

Find $\frac{d}{dx}\int_0^{2x+3}\left[(x+1)^2 + tx\right]dt$.

Determinant of a matrix

A *determinant* is a scalar quantity associated with a square matrix.

The determinant of a 2×2 matrix is equal to the product of the numbers on the leading diagonal (top left corner to bottom right corner) minus the product of the numbers on the other diagonal. It is written as $\det \mathbf{A}$, $|\mathbf{A}|$, or Δ when it is clear which matrix is involved.

> ***Example***
>
> What is $\det \mathbf{A}$ if $\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 4 & 3 \end{pmatrix}$?
>
> ***Solution***
>
> $\det \mathbf{A} = (2 \times 3) - (4 \times -6) = 30$

This is a specific definition, and we need to generalise. For any matrix $\mathbf{M}$, if c_{ij} is the element in the matrix corresponding to row *i* and column *j*, and M_{ij} is the determinant of the matrix formed when we strike out row *i* and column *j*, then the determinant is defined as $\sum_{j=1}^{n} (-1)^{j+1} c_{1j} M_{1j}$.

> ***Example***
>
> Find the determinant of $\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -4 \\ 2 & -3 & 1 \end{pmatrix}$.
>
> ***Solution***
>
> The determinant is given by:
>
> $$\Delta = 1 \times \begin{vmatrix} 0 & -4 \\ -3 & 1 \end{vmatrix} + (-1) \times (-1) \times \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix}$$
>
> $$= 1(0-12) + 1(3+8) + 2(-9-0) = -12 + 11 - 18 = -19$$

Eigenvectors and eigenvalues

There is an important aspect of matrix theory that arises from the fact that matrices can be used to represent transformations. If a vector **v** is transformed by a matrix **A**, then the resulting vector is **Av**. If the direction of the vector is unchanged once it has been transformed we can write:

$$\mathbf{Av} = \lambda\mathbf{v}$$

where λ is a constant. The vector **v** is called an *eigenvector* of the matrix, and the corresponding value of λ is called an *eigenvalue*.

To find the eigenvectors and eigenvalues, you have to work with the equation $\mathbf{Av} = \lambda\mathbf{v}$:

$$\mathbf{Av} = \lambda\mathbf{v} \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

where **0** is the zero matrix. Notice that we had to insert the identity matrix **I** into the equation since we cannot subtract a scalar from a matrix.

Thinking of our ways of solving "ordinary" equations, we see that this equation would be true if either $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$, or $\mathbf{v} = \mathbf{0}$, but these are not going to be very helpful since this is just the trivial solution and is not the one we are interested in.

To solve a general matrix equation $\mathbf{Bx} = \mathbf{C}$, we would normally find the inverse of **B** and calculate $\mathbf{x} = \mathbf{B}^{-1}\mathbf{C}$. Using this technique to try to solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, then if $(\mathbf{A} - \lambda\mathbf{I})$ has an inverse we get $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0}$, *ie* $\mathbf{v} = \mathbf{0}$. Since we are looking for a non-trivial solution, we must prevent this method from working. The only thing that would stop us getting $\mathbf{v} = \mathbf{0}$ would be if $\mathbf{A} - \lambda\mathbf{I}$ does not have an inverse *ie* it is singular. Remembering that singular matrices have a determinant of zero this gives us a way to find the eigenvalues and eigenvectors.

In summary, to find the eigenvalues of a matrix **A** we must solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Once you have the eigenvalues, you can return to the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, to find the eigenvectors.

The equation obtained from $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the *characteristic equation of the matrix*.